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On the dispersion of fluid particles

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The dispersion of fluid particles in a turbulent flow is described by transition-probability density functions. A renormalized expansion is made to establish the evolution equations of these functions. The resulting equations are nonlinear integrodifferential equations written in terms of Eulerian velocity-correlation functions. For the dispersion of a single particle, the equation at the zeroth order is the same as the one obtained by Roberts (1961). For the relative dispersion of a pair of particles, the equation is more convenient for applications than those of other theories. With this equation, Richardson's $\frac{4}{3}$-power law for relative diffusion is recovered analytically based upon the Kolmogoroff spectrum and numerically based upon the von Kármán spectrum and a smoothed experimental spectrum.

1. Introduction

The dispersion of fluid particles is the most fundamental problem in the study of turbulent diffusion. Started by Taylor (1921), this problem has been attacked mainly by two approaches. One approach follows Taylor. The moments of the displacement of a particle are related kinematically to the Lagrangian velocity-correlation functions. Because Eulerian correlation functions are more conveniently measured, the basic problem of this approach is the Lagrangian–Eulerian transformation of the velocity-correlation functions. The other approach uses transition-probability density functions, or briefly transition functions, to describe the dispersion. The main task of the second approach is to establish the evolution equations for the transition functions and, from a practical standpoint, to express the equations in terms of Eulerian velocity-correlation functions. It may be noted that, according to the independence approximation (Corrsin 1959), the Lagrangian–Eulerian transformation requires the determination of a weighting function, which is equivalent to the transition function (Weinstock 1976).

The formulation for the transition functions was first discussed by Batchelor (1949, 1952). Since the 1960s, interesting results for the evolution equations have been obtained by Deissler (1961), Roberts (1961), Kraichnan (1966), Saffman (1969), Knobloch (1977) and Lundgren (1981). These equations are generally in an integro-differential form and are closed by a truncation of a series of correlations. These series vary in form with the expansion method. Whether or not the truncation provides a good approximation then depends on the behaviour of the specific series.

Recently, remarkable developments in turbulence theory have been made by means of expansions in terms of response functions or propagators. In this respect, we may mention the series of articles by Kraichnan (1959, 1965, 1977*a*, *b*), Bourret 1962, 1965), Dupree (1966, 1972), Weinstock (1969, 1977). Misguich & Balescu (1975, 1982) and Tchen (1984*a*, *b*). According to Weinstock (1969), the mean response functions used by Kraichnan, in the direct-interaction approximation (DIA) and in the Lagrangian-history direct-interaction approximation (LHDI), and the coherent response function introduced by Dupree (1966, 1972) are equivalent to mean propagators. In the same article, Weinstock introduced an effective propagator, which determines turbulent transport. The effective propagator may be expanded in terms of either a 'bare' propagator known as the quasilinear expansion, which implies advection by the mean velocity, or in terms of the mean propagator known as the renormalized expansion (Misguich & Balescu 1975). Kraichnan (1977*a*, *b*) expanded the mean propagator in terms of the bare propagator, which is also a kind of renormalized expansion.

The present paper uses the mean propagator to derive evolution equations for the transition functions. Instead of the iteration formulas given by Weinstock (1969), we follow the considerations of Tchen (1984*a*, *b*) to first expand the correlation, containing Weinstock's effective propagator, in terms of the exact propagator. Then the resulting series is re-arranged and re-expanded in a series in terms of the mean propagator. The closure is made by a low-order truncation of the final series. However, a discussion is given for the behaviour of the entire series. It is shown that, rather than the weak-turbulence and the weak-coupling limits, the approximation works with low-order moments of the displacements of the particles and a parameter. It may be noted that our expansion is similar to that of Marcuvitz (1973) for the leading terms.

For the dispersion of a single particle, the equation at zeroth order is essentially the same as the one obtained by Roberts (1961). However, it is explained that this equation should be modified by corrections when high-order moments of the displacement are of interest, especially at short times. For the relative dispersion of a pair of particles, the resulting equation is different from those of other theories. Since only Eulerian velocity-correlation functions are involved, it is convenient for practical applications. After Fourier transformation, this equation shows the dominance of eddies of scales comparable to the effective separation of the two particles. Based upon the Kolmogoroff spectrum, the analysis shows a τ^3 behaviour of the variance of the relative displacement, which corresponds to the $\frac{4}{3}$-power law of relative diffusion discovered by Richardson (1926). Numerical calculations are made based upon the von Kármán spectrum and a spectrum from an experiment in atmospheric turbulence. The τ^3 behaviour is also found.

2. Renormalized expansion

2.1. *Transition functions*

The turbulent fluid flow is assumed to be incompressible, statistically stationary and homogeneous. Molecular diffusivity will be neglected. Let the present position of a 'marked' fluid particle be $\hat{\boldsymbol{X}}(t)$ and its initial position be $\boldsymbol{x}_0$. During the time period $\tau = t - t_0$ the displacement of the particle is $\hat{\boldsymbol{Y}}(\tau) = \hat{\boldsymbol{X}}(t) - \boldsymbol{x}_0$. The velocity of the particle can then be expressed as

$$\hat{\boldsymbol{v}}(\tau) = \frac{\mathrm{d}\hat{\boldsymbol{Y}}(\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}\hat{\boldsymbol{X}}(t)}{\mathrm{d}t}, \tag{1}$$

where the t_0 dependence of $\hat{\boldsymbol{v}}(\tau)$ is understood. The symbol $(\hat{\ })$ is used for a fluctuating quantity. The symbols $(\bar{\ }) = \langle\ \rangle$ and $(\tilde{\ })$ will be used to represent the ensemble average and the fluctuation of the quantity. The micro-distribution of the displacement can be specified by a delta function,

$$\hat{P}(\boldsymbol{y}, \tau \mid \boldsymbol{x}_0, t_0) = \delta[\boldsymbol{y} - \hat{\boldsymbol{Y}}(\tau)], \tag{2}$$

which may be called the instantaneous transition function. The ensemble average of $\hat{P}(\boldsymbol{y},\tau\,|\,\boldsymbol{x}_0,t_0)$, $\bar{P}(\boldsymbol{y},\tau)$, is the probability density function for the particle to make the transition between the two states $(\boldsymbol{x}_0,t_0)$ and $(\boldsymbol{x},t)$, with $\boldsymbol{y}=\boldsymbol{x}-\boldsymbol{x}_0$. For stationary and homogeneous turbulence, $\bar{P}(\boldsymbol{y},\tau)$ does not depend on the initial condition $(\boldsymbol{x}_0,t_0)$. $\bar{P}(\boldsymbol{y},\tau)$ will be called the one-particle transition function for brevity. The partial derivative of (2) with respect to τ leads to the equation

$$[\partial_\tau+\hat{\boldsymbol{v}}(\tau)\cdot\boldsymbol{\nabla}]\,\hat{P}(\boldsymbol{y},\tau\,|\,\boldsymbol{x}_0,t_0)=0, \tag{3}$$

where $\partial_\tau\equiv\partial/\partial\tau$ and $\boldsymbol{\nabla}\equiv\partial/\partial\boldsymbol{y}$. For a fluid particle, its velocity is the same as the local Eulerian velocity, so that (3) is sometimes called the convective equation. Define the differential operator $\hat{\mathrm{L}}(\tau)\equiv\hat{\boldsymbol{v}}(\tau)\cdot\boldsymbol{\nabla}$. Then (3) is turned into the Liouville form

$$[\partial_\tau+\hat{\mathrm{L}}(\tau)]\,\hat{P}(\tau)=0, \tag{4}$$

where the spatial dependence of $\hat{P}$ is implied. The evolution equation of the transition function $\bar{P}$ is to be established by averaging (4) over all realizations.

Similarly to (54), the equation for a pair of marked fluid particles can be written as

$$[\partial_\tau+\hat{\mathrm{L}}_{12}(\tau)]\,\hat{P}_{12}(\tau)=0, \tag{5}$$

with
$$\hat{\mathrm{L}}_{12}(\tau)\equiv\hat{\mathrm{L}}_1(\tau)+\hat{\mathrm{L}}_2(\tau)=\hat{\boldsymbol{v}}_1(\tau)\cdot\boldsymbol{\nabla}_1+\hat{\boldsymbol{v}}_2(\tau)\cdot\boldsymbol{\nabla}_2,$$

and
$$\begin{aligned}\hat{P}_{12}(\tau)&\equiv\hat{P}_{12}(\boldsymbol{y}_1,\boldsymbol{y}_2,\tau\,|\,\boldsymbol{x}_{10},\boldsymbol{x}_{20},t_0)\\&\equiv\hat{P}_1(\boldsymbol{y}_1,\tau\,|\,\boldsymbol{x}_{10},t_0)\,\hat{P}_2(\boldsymbol{y}_2,\tau\,|\,\boldsymbol{x}_{20},t_0).\end{aligned}$$

The subscripts 1 and 2 are used to distinguish the two particles. The ensemble average of $\hat{P}_{12}$ is the two-particle transition function for the two states $(\boldsymbol{x}_{10},\boldsymbol{x}_{20},t_0)$ and $(\boldsymbol{x}_1,\boldsymbol{x}_2,t)$, with $\boldsymbol{y}_1=\boldsymbol{x}_1-\boldsymbol{x}_{10}$ and $\boldsymbol{y}_2=\boldsymbol{x}_2-\boldsymbol{x}_{20}$. For stationary and homogeneous turbulence, $\bar{P}_{12}$ does not depend on either $\boldsymbol{x}_{10}$ or $\boldsymbol{x}_{20}$, but on the initial separation $\boldsymbol{r}_0=\boldsymbol{x}_{20}-\boldsymbol{x}_{10}$. Therefore it is written as

$$\bar{P}_{12}(\boldsymbol{y}_1,\boldsymbol{y}_2,\tau\,|\,\boldsymbol{r}_0,t_0)=\langle\hat{P}_{12}(\boldsymbol{y}_1,\boldsymbol{y}_2,\tau\,|\,\boldsymbol{x}_{10},\boldsymbol{x}_{20},t_0)\rangle=\bar{P}_{12}(\boldsymbol{y}_1,\boldsymbol{y}_1+\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0),$$

where $\boldsymbol{\lambda}$ refers to the relative displacement such that

$$\hat{\boldsymbol{\lambda}}(\tau)=\hat{\boldsymbol{Y}}_2(\tau)-\hat{\boldsymbol{Y}}_1(\tau).$$

Then the relative transition function, specifying the relative displacement, is defined by

$$\bar{B}(\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0)=\int\mathrm{d}\boldsymbol{y}_1\,\bar{P}_{12}(\boldsymbol{y}_1,\boldsymbol{y}_1+\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0). \tag{6}$$

General properties of these transition functions and discussions based upon dimensional reasoning can be found in Batchelor (1952), Roberts (1961) and Monin & Yaglom (1975). The following properties will be used frequently:

$$\int\mathrm{d}\boldsymbol{y}\,\bar{P}(\boldsymbol{y},\tau)=1, \tag{7}$$

$$\int\mathrm{d}\boldsymbol{\lambda}\,\bar{P}_{12}(\boldsymbol{y}_1,\boldsymbol{y}_1+\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0)=\bar{P}_1(\boldsymbol{y}_1,\tau), \tag{8}$$

$$\iint\mathrm{d}\boldsymbol{y}_1\,\mathrm{d}\boldsymbol{\lambda}\,\bar{P}_{12}(\boldsymbol{y}_1,\boldsymbol{y}_1+\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0)=1, \tag{9}$$

$$\int\mathrm{d}\boldsymbol{\lambda}\,\bar{B}(\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0)=1. \tag{10}$$

The trajectories of the two particles are shown in figure 1.

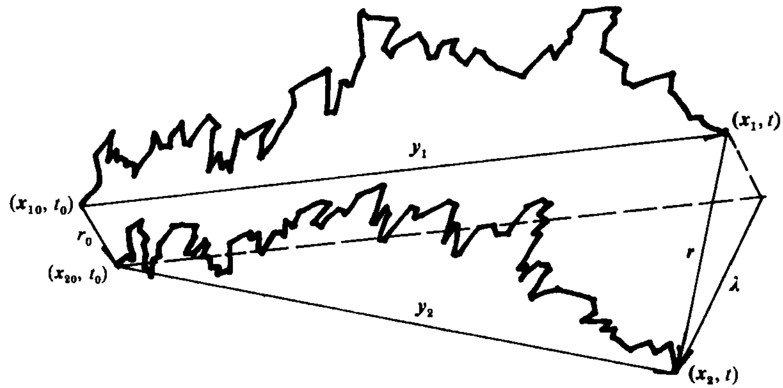

FIGURE 1. Trajectories of a pair of particles.

2.2. *Renormalized expansion*

The ensemble average of (4) yields

$$(\partial_\tau + \overline{\mathrm{L}})\,\bar{P} = \bar{C} \equiv -\langle \tilde{\mathrm{L}}\tilde{P}\rangle, \tag{11}$$

where $\tilde{\mathrm{L}}$ and $\tilde{P}$ are the fluctuations of $\hat{\mathrm{L}}$ and $\hat{P}$ respectively. The evolution equation of $\bar{P}$ is to be established by expressing $\bar{C}$ explicitly in terms of $\bar{P}$ and velocity-correlation functions of the fluid flow. For this purpose, (11) is subtracted from (4) to give

$$(\partial_\tau + \hat{\mathrm{L}})\,\tilde{P} = -\tilde{\mathrm{L}}\bar{P} + \langle \tilde{\mathrm{L}}\tilde{P}\rangle. \tag{12}$$

The procedure is to solve for $\tilde{P}$ from (12) and to substitute the result into $\bar{C}$.

The exact propagator is introduced for (12), defined by

$$\left.\begin{aligned} [\partial_\tau + \hat{\mathrm{L}}(\tau)]\,\hat{U}(\tau,\tau_0) &= 0,\\ \hat{U}(\tau_0,\tau_0) &= 1,\end{aligned}\right\} \tag{13}$$

and is called the exact propagator because the velocity contained in $\hat{\mathrm{L}}$ is the exact velocity possessed by the particle (Weinstock 1969). Note that τ refers to the transition time $t-t_0$, so that $\tau_0 = 0$. $\tilde{P}$ is then formally found from (12) and substituted into (11), yielding

$$\bar{C} = \langle \tilde{\mathrm{L}}\hat{U} * \tilde{\mathrm{L}}\rangle\,\bar{P} - \langle \tilde{\mathrm{L}}\hat{U}\rangle * \langle \tilde{\mathrm{L}}\tilde{P}\rangle, \tag{14}$$

where the star is used to represent a time integration, e.g.

$$\langle \tilde{\mathrm{L}}\hat{U} * \tilde{\mathrm{L}}\rangle\,\bar{P} \equiv \int_0^\tau \mathrm{d}\tau' \langle \tilde{\mathrm{L}}(\tau)\,\hat{U}(\tau,\tau')\,\tilde{\mathrm{L}}(\tau')\rangle\,\bar{P}(\tau'),$$

and the initial fluctuation $\tilde{P}(0)$ is neglected as usual. The mean propagator is introduced as the deterministic part of $\hat{U}$, $\bar{U} = \langle \hat{U}\rangle$, to get rid of the stochasticity in $\hat{U}$. Equation (12) can be written in other forms. Accordingly, U^0, the bare propagator, and $\hat{\Lambda}$, the effective propagator (Weinstock 1969), can be introduced. Weinstock derived iteration formulas relating $\hat{\Lambda}$ and $\hat{U}$ to U^0, $\bar{U}$ and $\hat{U}$. However, with the effective propagator, (14) is written as

$$\bar{C} = \langle \tilde{\mathrm{L}}\hat{\Lambda} * \tilde{\mathrm{L}}\rangle\,\bar{P}, \tag{15}$$

which is of major interest. Following Tchen (1984*a*, *b*), we first expand (14) in terms of $\hat{U}$. Then the resulting series is rearranged and re-expanded in terms of $\bar{U}$ only.

Essentially the series is an expansion of the correlation with $\hat{A}$, (15), but not $\hat{A}$ itself. The procedure and the main results are as follows.

As in (11) and (12), one can obtain equations for $\overline{U}$ and $\tilde{U}$ by averaging (13):

$$(\partial_\tau + \overline{\mathrm{L}})\,\overline{U} = \overline{H} \equiv -\langle \tilde{\mathrm{L}}\tilde{U}\rangle, \tag{16}$$

$$(\partial_\tau + \tilde{\mathrm{L}})\,\tilde{U} = -\tilde{\mathrm{L}}\overline{U} + \langle \tilde{\mathrm{L}}\tilde{U}\rangle, \tag{17}$$

such that

$$\tilde{U} = -\hat{U} * \tilde{\mathrm{L}}\overline{U} + \hat{U} * \langle \tilde{\mathrm{L}}\tilde{U}\rangle = -\overline{U} * \tilde{\mathrm{L}}\overline{U} - \tilde{U} * \tilde{\mathrm{L}}\overline{U} + \overline{U} * \langle \tilde{\mathrm{L}}\tilde{U}\rangle + \tilde{U} * \langle \tilde{\mathrm{L}}\tilde{U}\rangle, \tag{18}$$

$$\overline{H} = \langle \tilde{\mathrm{L}}\hat{U} * \tilde{\mathrm{L}}\rangle\,\overline{U} - \langle \tilde{\mathrm{L}}\tilde{U}\rangle * \langle \tilde{\mathrm{L}}\tilde{U}\rangle. \tag{19}$$

Define $\qquad \overline{C}_{\mathrm{s}} \equiv \langle \tilde{\mathrm{L}}\hat{U} * \tilde{\mathrm{L}}\rangle\,\overline{P}, \quad \overline{H}_{\mathrm{s}} \equiv \langle \tilde{\mathrm{L}}\hat{U} * \tilde{\mathrm{L}}\rangle\,\overline{U}.$

Then (14) and (19) can be written as

$$\overline{C} = \overline{C}_{\mathrm{s}} - \overline{H} * \overline{C}, \tag{20}$$

$$\overline{H} = \overline{H}_{\mathrm{s}} - \overline{H} * \overline{H}. \tag{21}$$

These two equations are analogous to those given by Tchen (1984*a, b*) using a kinetic approach. By means of (20) and (21), $\overline{C}$ is expanded in terms of $\overline{H}_{\mathrm{s}}$ and $\overline{C}_{\mathrm{s}}$:

$$\overline{C} = \overline{C}_{\mathrm{s}} + \sum_{i=1}^{\infty} \frac{(-1)^i\, 2^i (2i-1)!!}{(i+1)!} (\overline{H}_{\mathrm{s}} *)^i\, \overline{C}_{\mathrm{s}}, \tag{22}$$

with the condition that the norm of $\overline{H}_{\mathrm{s}} *$ satisfies

$$\| \overline{H}_{\mathrm{s}} * \| = \| \langle \tilde{\mathrm{L}}\hat{U} * \tilde{\mathrm{L}}\rangle\,\overline{U} * \| < \tfrac{1}{4}. \tag{23}$$

This condition is estimated in a way similar to that used by Knobloch (1978), and is given explicitly in §2.3. Next, $\overline{C}_{\mathrm{s}}$ and $\overline{H}_{\mathrm{s}}$ are written as

$$\overline{C}_{\mathrm{s}} = \overline{C}_0 + \langle \tilde{\mathrm{L}}\tilde{U} * \tilde{\mathrm{L}}\rangle\,\overline{P}, \quad \overline{H}_{\mathrm{s}} = \overline{H}_0 + \langle \tilde{\mathrm{L}}\tilde{U} * \tilde{\mathrm{L}}\rangle\,\overline{U},$$

with $\qquad \overline{C}_0 \equiv \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{P}, \quad \overline{H}_0 \equiv \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{U}.$

Rearrange series (22) with the new forms of $\overline{C}_{\mathrm{s}}$ and $\overline{H}_{\mathrm{s}}$. The resulting series appears in a form which is conveniently expanded to get the final expansion of $\overline{C}$ in terms of $\overline{U}$ only. Unfortunately, unlike the series in (22), the general term has not yet been found. The leading terms are

$$\overline{C} = \overline{C}_0 + \overline{C}_3 + \overline{C}_4 + \dots, \tag{24}$$

with

$$\overline{C}_0 = \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{P},$$
$$\overline{C}_3 = -\langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{P},$$
$$\overline{C}_4 = \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{P} - \langle \tilde{\mathrm{L}}\overline{U} * \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{P} - \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{U} * \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\rangle\,\overline{P},$$

where the subscript of $\overline{C}_n$ refers to the number of $\tilde{\mathrm{L}}$ contained in the term. $\overline{C}_0$ has a second-order correlation, but refers to the zeroth-order approximation. Higher-order terms are very complicated. The terms up to the eighth order are given by Jiang (1984).

2.3. *Closure*

Like the typical problem in turbulence study, the series (24) implies a hierarchy of correlations of increasing order. The closure is made by an arbitrary truncation. Nevertheless, as mentioned by other authors, the renormalized expansion produces a series that behaves better than the one obtained by the quasilinear expansion. The following discussion is given to estimate the conditions for a low-order truncation.

In the short-time limit the velocity of the marked fluid particle is approximately the same as its initial velocity. With the condition that the turbulence is stationary and homogeneous, it can be easily shown that all the odd-order moments of the displacement of the particle are approximately zero and that an even-order moment, say the $2n$th, is contributed by the even-order terms in (24) up to the $2n$th. It can also be shown that, although smaller, all these even terms contribute to the $2n$th moment with the same order as that of $\overline{C}_0 = \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}} \rangle \overline{P}$. By comparison of the result with that of the kinematics of the particle, it can be shown that a low-order truncation is not good when high-order moments are of interest, no matter how weak the turbulence is. Therefore a low-order truncation, at short times, only applies to the case where low-order moments are to be investigated.

At long times, several approximations may be applied to simplify the series (24). First, the correlations in (24) contain multiple space–time points. In the space–time integrations implied in the operations of $\overline{U} *$, one or several points could be away from other points with a high probability. Therefore the odd-order terms may be neglected in comparison with the even-order terms. Secondly, the series (24) contains correlations in all possible forms. The terms containing 'non-neighbouring' correlations may be neglected because of the restriction of the correlation time and length. For example, $\langle \tilde{\mathrm{L}}\overline{U} * \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}} \rangle \overline{U} * \tilde{\mathrm{L}} \rangle \overline{P}$ has a correlation separated by another correlation of the fourth order. It may be neglected in comparison with $\langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}} \rangle \overline{U} * \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}} \rangle \overline{U} * \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}} \rangle \overline{P}$. Since the full expression for (24) is very complicated and no general form of the terms has yet been found, a further approximation is needed. The terms up to the eighth order are reduced by the first two approximations. A general term can be found from these reduced terms. We then assume that the general term applies to the whole series. The result is

$$\overline{C} = \sum_{i=1}^{\infty} \overline{C}_{2i}, \tag{25}$$

with

$$\left.\begin{aligned} \overline{C}_2 &\equiv \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}} \rangle \overline{P}, \\ \hat{C}_2 &\equiv \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{P}, \\ \overline{C}_{2(i+1)} &\approx \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{U} * (\hat{C}_{2i} - \overline{C}_{2i}) \rangle \quad (i = 1, 2, 3, \ldots). \end{aligned}\right\} \tag{26}$$

The norm of $\tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}}\overline{U} *$ is approximately the same as that of $\overline{H}_{\underline{s}} *$. However, because of the outer brackets (ensemble average) of the common term $\overline{C}_{2(i+1)}$, the series (25) converges faster than does the series (22) when condition (23) is satisfied. Therefore at long times a low-order truncation of (24), or (25), is a good approximation.

A discussion for intermediate times does not seem possible at present. In the rest of this paper, we assume that a low-order truncation applies to all times based upon the discussion of the short-time and long-time cases and the fact that low-order moments of the displacement of the particle are of the most practical interest. For the second moment, the resulting equation is

$$(\partial_\tau + \tilde{\mathrm{L}}) \overline{P} = \langle \tilde{\mathrm{L}}\overline{U} * \tilde{\mathrm{L}} \rangle \overline{P}. \tag{27}$$

Correction terms should be added when higher-order moments are involved. The third-order correlation, as a correction, was discussed by Weinstock (1976) for a related subject, the Lagrangian–Eulerian transformation of velocity-correlation functions. However, for the evolution equation of the transition function, $\overline{C}_4$ is more important than $\overline{C}_3$. We suggest that the first correction is

$$\begin{aligned}\overline{C}_4 &= \langle \tilde{L}\overline{U} * \tilde{L}\overline{U} * \tilde{L}\overline{U} * \tilde{L}\rangle \overline{P} - \langle \tilde{L}\overline{U} * \langle \tilde{L}\overline{U} * \tilde{L}\rangle \overline{U} * \tilde{L}\rangle \overline{P} - \langle \tilde{L}\overline{U} * \tilde{L}\rangle \overline{U} * \langle \tilde{L}\overline{U} * \tilde{L}\rangle \overline{P} \\ &= \langle \tilde{L}\overline{U} * \tilde{L}\overline{U} * \tilde{L}\overline{U} * \tilde{L}\rangle (1,3;2,4)\, \overline{P},\end{aligned} \tag{28}$$

where the symbol $(1,3;2,4)$ means that the fourth-order correlation is decoupled into two pair-correlations, the first $\tilde{L}$ with the third and the second $\tilde{L}$ with the fourth. The second step of (28) is made so that the second moment of the Eulerian velocity field can be applied, similar to the approximation made by Bourret (1962).

The equation for the transition function of a pair of marked fluid particles can by analogy be obtained from (5). The result is, for the second moments of the displacements,

$$(\partial_\tau + \overline{L}_{12})\, \overline{P}_{12} = \sum_{i,j=1,2} \langle \tilde{L}_i \overline{U}_{12} * \tilde{L}_j \rangle \overline{P}_{12}, \tag{29}$$

where $\overline{U}_{12}$ is the mean propagator of the two particles, respective to the two-particle exact propagator.

Equations (27) and (29) are not restricted by the weak-turbulence and the weak-coupling limits. The condition can be estimated by means of (23). The operation of $\overline{U}\,*$ implies a temporal integration, and the spatial derivative in $\tilde{L}$ operates on $\overline{P}$, which is implied in $\overline{U}$. The condition (23) is then estimated by the magnitudes,

$$R \equiv \| \overline{H}_s * \|^{\frac{1}{2}} \equiv \langle \tilde{u}^2 \rangle^{\frac{1}{2}} \mathscr{T}_L / \mathscr{L}_L < \tfrac{1}{2}, \tag{30}$$

where $\langle \tilde{u}^2 \rangle^{\frac{1}{2}}$ is the r.m.s. velocity, $\mathscr{T}_L$ is a Lagrangian time-scale and $\mathscr{L}_L$ is a Lagrangian lengthscale over which, from the present position of the particle, $\overline{P}$ has a significant change. This condition is comparable to that suggested by Knobloch (1978).

3. Dispersion of fluid particles

3.1. *Dispersion of a single particle*

The exact propagator restricts its operand to be evaluated along the trajectory of the particle. Such an operation can be mathematically replaced by a spatial integration weighted by a delta function. By means of (2), we can write

$$\hat{U}(\tau,\tau')\, F(\tau') = \int \mathrm{d}\boldsymbol{y}'\, \hat{P}(\boldsymbol{y}-\boldsymbol{y}', \tau-\tau' \,|\, \boldsymbol{x}', t')\, F(\boldsymbol{y}', \tau'),$$

so that

$$\overline{U}(\tau,\tau')\, F(\tau') = \int \mathrm{d}\boldsymbol{y}'\, \overline{P}(\boldsymbol{y}-\boldsymbol{y}', \tau-\tau')\, F(\boldsymbol{y}', \tau'), \tag{31}$$

where F is a quantity carried by the particle. This equivalence is shown in detail in the Appendix. Recall that the velocity of the particle is the same as the local Eulerian velocity. By means of relation (31), the evolution equation (27) for a single particle is written explicitly as

$$(\partial_\tau + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\, \overline{P}(\boldsymbol{y},\tau) = \boldsymbol{\nabla}\cdot \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}\boldsymbol{y}'\, \boldsymbol{R}(\boldsymbol{y}-\boldsymbol{y}', \tau-\tau')\, \overline{P}(\boldsymbol{y}-\boldsymbol{y}', \tau-\tau')\cdot \boldsymbol{\nabla}'\, \overline{P}(\boldsymbol{y}', \tau'). \tag{32}$$

In (32), $\bar{\boldsymbol{u}}$ is the mean velocity, $\boldsymbol{\nabla} = \partial/\partial \boldsymbol{y}$, $\boldsymbol{\nabla}' = \partial/\partial \boldsymbol{y}'$ and

$$\boldsymbol{R}(\boldsymbol{y}-\boldsymbol{y}', \tau-\tau') = \langle \tilde{\boldsymbol{u}}(\boldsymbol{y}+\boldsymbol{x}_0, \tau+t_0)\, \tilde{\boldsymbol{u}}(\boldsymbol{y}'+\boldsymbol{x}_0, \tau'+t_0)\rangle$$

is the Eulerian velocity-correlation function. The correction term (28) is found as

$$
\begin{aligned}
\bar{C}_4 = \nabla_i \int_0^{\tau} \mathrm{d}\tau' \int_0^{\tau'} \mathrm{d}\tau'' \int_0^{\tau''} \mathrm{d}\tau''' \iiint \mathrm{d}y'\,\mathrm{d}y''\,\mathrm{d}y''' \\
\times R_{im}(y-y'', \tau-\tau'')\, R_{jn}(y'-y''', \tau'-\tau''')\, \bar{P}(y-y', \tau-\tau') \\
\times \nabla'_j \bar{P}(y'-y'', \tau'-\tau''')\, \nabla''_m \bar{P}(y''-y''', \tau''-\tau''')\, \nabla'''_n \bar{P}(y''', \tau'''),
\end{aligned}
\tag{33}
$$

where the subscripts refer to the components of the respective vectors. The summation convention in (33) will be used throughout the paper unless noted. Equation (32) is essentially the same as the one obtained by Roberts (1961), so that only some remarks will be given. General properties of the one-particle dispersion can be found in the paper by Roberts.

The second moment of the displacement can be found to obey

$$
\frac{1}{2}\frac{\mathrm{d}\langle \boldsymbol{Y}(\tau)\, \boldsymbol{Y}(\tau)\rangle}{\mathrm{d}\tau} = \int_0^{\tau} \mathrm{d}\tau' \int \mathrm{d}y'\, \boldsymbol{R}(y', \tau')\, \bar{P}(y', \tau') \tag{34}
$$

by means of (32). On the other hand, as considered by Roberts one may apply a memory cutoff to (32) to get a diffusion-like equation. Replace $\boldsymbol{\nabla}'\bar{P}(y', \tau')$ by $\boldsymbol{\nabla}\bar{P}(y, \tau)$. The result is

$$
(\partial_\tau + \bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla})\, \bar{P}(y, \tau) \approx \boldsymbol{\nabla}\cdot\boldsymbol{K}(\tau)\cdot\boldsymbol{\nabla}\bar{P}(y, \tau). \tag{35}
$$

The interesting point is that the expression for the eddy-diffusivity tensor $\boldsymbol{K}(\tau)$ is exactly the same as (34). It is well known that one-particle dispersion is nearly Gaussian at all times and Taylor's (1921) formula can be used as the eddy diffusivity. Equations (34) and (35) provide added support for the Gaussian approximation. The Fourier transform of (34) for isotropic turbulence and the Gaussian approximation give

$$
\frac{\mathrm{d}\sigma^2(\tau)}{\mathrm{d}\tau} = \frac{4}{3}\int_0^{\tau} \mathrm{d}\tau' \int_0^{\infty} \mathrm{d}k\, E(k, \tau') \exp\left[-\tfrac{1}{2}k^2\sigma^2(\tau')\right], \tag{36}
$$

where $\sigma^2(\tau)$ is the one-dimensional variance of $\bar{P}(y, \tau)$. This equation will be used later. Here and after, any function, with the argument k, refers to its Fourier transform.

The correction (33) should be added when higher-order moments of $\bar{P}$ are involved, such as in discussing the flatness of the profile of $\bar{P}$. It can be shown that with (33) the fourth cumulant of $\bar{P}$ is approximately zero in the short-time limit and at long times. The departure of $\bar{P}$ from Gaussian happens mainly at short times. The profile of $\bar{P}$ is generally steeper than that of a Gaussian approximation, and the vorticity spectrum plays an important role (Jiang 1984).

3.2. *Relative dispersion of a pair of particles*

In a similar way to the one-particle case, the two-particle mean propagator can be related to the two-particle transition function:

$$
\bar{U}_{12}(\tau, \tau')\, F(\tau') = \iint \mathrm{d}y'_1\, \mathrm{d}y'_2\, \bar{P}_{12}(y_1 - y'_1, y_2 - y'_2, \tau - \tau' \,|\, r', t')\, F(y'_1, y'_2, \tau').
$$

By means of this relation, (29) can be expressed explicitly in terms of Eulerian velocity-correlation functions. The result is

$$\begin{aligned}
&(\partial_\tau + \bar{\boldsymbol{u}}\cdot\nabla_1 + \bar{\boldsymbol{u}}\cdot\nabla_2)\bar{P}_{12}(\boldsymbol{y}_1, \boldsymbol{y}_2, \tau \mid \boldsymbol{r}_0, t_0)\\
&= \nabla_1\cdot\int_0^\tau d\tau' \iint d\boldsymbol{y}_1' \, d\boldsymbol{y}_2' \, \boldsymbol{R}(\boldsymbol{y}_1 - \boldsymbol{y}_1', \tau-\tau')\\
&\quad \times \bar{P}_{12}(\boldsymbol{y}_1 - \boldsymbol{y}_1', \boldsymbol{y}_2 - \boldsymbol{y}_2', \tau-\tau' \mid \boldsymbol{r}', t')\cdot\nabla_1' \bar{P}_{12}(\boldsymbol{y}_1', \boldsymbol{y}_2', \tau' \mid \boldsymbol{r}_0, t_0)\\
&\quad \nabla_1\cdot\int_0^\tau d\tau' \iint d\boldsymbol{y}_1' \, d\boldsymbol{y}_2' \, \boldsymbol{R}(\boldsymbol{y}_1 - \boldsymbol{y}_2' - \boldsymbol{r}_0, \tau-\tau')\\
&\quad \times \bar{P}_{12}(\boldsymbol{y}_1 - \boldsymbol{y}_1', \boldsymbol{y}_2 - \boldsymbol{y}_2', \tau-\tau' \mid \boldsymbol{r}', t')\cdot\nabla_2' \bar{P}_{12}(\boldsymbol{y}_1', \boldsymbol{y}_2', \tau' \mid \boldsymbol{r}_0, t_0)\\
&\quad + \nabla_2\cdot\int_0^\tau d\tau' \iint d\boldsymbol{y}_1' \, d\boldsymbol{y}_2' \, \boldsymbol{R}(\boldsymbol{y}_2 - \boldsymbol{y}_1' + \boldsymbol{r}_0, \tau-\tau')\\
&\quad \times \bar{P}_{12}(\boldsymbol{y}_1 - \boldsymbol{y}_1', \boldsymbol{y}_2 - \boldsymbol{y}_2', \tau-\tau' \mid \boldsymbol{r}', t')\cdot\nabla_1' \bar{P}_{12}(\boldsymbol{y}_1', \boldsymbol{y}_2', \tau' \mid \boldsymbol{r}_0, t_0)\\
&\quad + \nabla_2\cdot\int_0^\tau d\tau' \iint d\boldsymbol{y}_1' \, d\boldsymbol{y}_2' \, \boldsymbol{R}(\boldsymbol{y}_2 - \boldsymbol{y}_2', \tau-\tau')\\
&\quad \times \bar{P}_{12}(\boldsymbol{y}_1 - \boldsymbol{y}_1', \boldsymbol{y}_2 - \boldsymbol{y}_2', \tau-\tau' \mid \boldsymbol{r}', t')\cdot\nabla_2' \bar{P}_{12}(\boldsymbol{y}_1', \boldsymbol{y}_2', \tau' \mid \boldsymbol{r}_0, t_0),
\end{aligned} \tag{37}$$

where $\nabla_1 = \partial/\partial \boldsymbol{y}_1$ and $\nabla_1' = \partial/\partial \boldsymbol{y}_1'$. Transform (37) into the $(\boldsymbol{y}_1, \boldsymbol{\lambda})$-coordinate system and integrate the resulting equation with respect to $\boldsymbol{y}_1$ to get

$$\begin{aligned}
&\partial_\tau \bar{B}(\boldsymbol{\lambda}, \tau \mid \boldsymbol{r}_0, t_0)\\
&= \nabla_\lambda\cdot\int_0^\tau d\tau' \iint d\boldsymbol{y}' \, d\boldsymbol{\lambda}' \, [2\boldsymbol{R}(\boldsymbol{y}', \tau-\tau') - \boldsymbol{R}(\boldsymbol{y}' - \boldsymbol{\lambda}' - \boldsymbol{r}_0, \tau-\tau') - \boldsymbol{R}(\boldsymbol{y}' + \boldsymbol{\lambda}' + \boldsymbol{r}_0, \tau-\tau')]\\
&\qquad \times \bar{P}_{12}(\boldsymbol{y}', \boldsymbol{y}' + \boldsymbol{\lambda} - \boldsymbol{\lambda}', \tau-\tau' \mid \boldsymbol{r}', t')\cdot\nabla_\lambda' \bar{B}(\boldsymbol{\lambda}', \tau' \mid \boldsymbol{r}_0, t_0),
\end{aligned} \tag{38}$$

where $\nabla_\lambda \equiv \partial/\partial\boldsymbol{\lambda}$ and $\nabla_\lambda' \equiv \partial/\partial\boldsymbol{\lambda}'$. It should be noted that, although subscripted by 1 and 2, the two particles are not actually distinguishable. Therefore $\bar{P}_{12}(\boldsymbol{y}', \boldsymbol{y}' + \boldsymbol{\lambda} - \boldsymbol{\lambda}', \tau-\tau' \mid \boldsymbol{r}', t')$ and $\bar{P}_{12}(\boldsymbol{y}' + \boldsymbol{\lambda}' - \boldsymbol{\lambda}, \boldsymbol{y}', \tau-\tau' \mid -\boldsymbol{r}', t')$ can be treated as the same when they appear in the intermediate steps of the derivation, as the result of interchanging the two particles. Equation (38) is the equation governing the relative dispersion.

As mentioned by other authors (Batchelor 1952; Roberts 1961), the dispersion of a single particle is dominated by large eddies and the relative dispersion of a pair of particles is dominated by small eddies. Before the separation of the two particles becomes comparable to large scales, the two dispersion processes, which compose the two-particle dispersion, may be assumed independent of each other. Then (38) can be further reduced to

$$\begin{aligned}
&\partial_\tau \bar{B}(\boldsymbol{\lambda}, \tau \mid \boldsymbol{r}_0, t_0)\\
&= \nabla_\lambda\cdot\int_0^\tau d\tau' \iint d\boldsymbol{y}' \, d\boldsymbol{\lambda}' \, [2\boldsymbol{R}(\boldsymbol{y}', \tau-\tau') - \boldsymbol{R}(\boldsymbol{y}' - \boldsymbol{\lambda}' - \boldsymbol{r}_0, \tau-\tau') - \boldsymbol{R}(\boldsymbol{y}' + \boldsymbol{\lambda}' + \boldsymbol{r}_0, \tau-\tau')]\\
&\qquad \times \bar{P}(\boldsymbol{y}', \tau-\tau')\, \bar{B}(\boldsymbol{\lambda} - \boldsymbol{\lambda}', \tau-\tau' \mid \boldsymbol{r}', t')\cdot\nabla_\lambda' \bar{B}(\boldsymbol{\lambda}', \tau' \mid \boldsymbol{r}_0, t_0).
\end{aligned} \tag{39}$$

In principle, when the Eulerian velocity-correlation functions are given, $\bar{P}$ can be solved from (32), and then $\bar{B}$ can be solved from (39).

Equation (38) appears to be similar to the one obtained by Roberts (1961).

However, in our equation, $\bar{P}_{12}$ accounts for the coupling of the two trajectories, so that the main feature of the relative dispersion is retained. Since only Eulerian velocity-correlation functions are involved, our equation is convenient for practical applications. Like other renormalized expansions, the Galilean invariance can be shown to be satisfied, which was emphasized by Lundgren (1981) for the relative dispersion. It is interesting to note that a memory cutoff of (38) or (39) would lead to an equation having the same physical meaning as that of the equation derived by Kraichnan (1966). However, for the case that the two particles are initially close to each other, the memory effect is strong. Therefore the memory cutoff will not be considered.

3.3. *Properties of relative dispersion and Richardson's $\frac{4}{3}$-power law*

In the short-time limit (38) can be approximately written as

$$\partial_\tau \bar{B}(\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0) \approx [2\boldsymbol{R}(0,0)-\boldsymbol{R}(-\boldsymbol{r}_0,0)-\boldsymbol{R}(\boldsymbol{r}_0,0)]:\nabla_\lambda \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}\boldsymbol{\lambda}' \times \bar{B}(\boldsymbol{\lambda}-\boldsymbol{\lambda}',\tau-\tau'\,|\,\boldsymbol{r}',t')\,\nabla'_\lambda B(\boldsymbol{\lambda}',\tau'\,|\,\boldsymbol{r}_0,t_0).$$

By means of this equation, the second moment of the relative displacement of the two particles is found to obey

$$\frac{\mathrm{d}\langle\tilde{\boldsymbol{\lambda}}(\tau)\,\tilde{\boldsymbol{\lambda}}(\tau)\rangle}{\mathrm{d}\tau} = \int \mathrm{d}\boldsymbol{\lambda}\,\boldsymbol{\lambda}\boldsymbol{\lambda}\,\bar{B}(\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0) \approx 2[2\boldsymbol{R}(0,0)-\boldsymbol{R}(-\boldsymbol{r}_0,0)-\boldsymbol{R}(\boldsymbol{r}_0,0)].$$

Recall that $\boldsymbol{r} = \boldsymbol{\lambda}+\boldsymbol{r}_0$. The variance of the separation of the two particles is found as

$$\begin{aligned}\langle\tilde{r}^2(\tau)\rangle &= r_0^2+\langle\tilde{\lambda}^2(\tau)\rangle \\ &\approx r_0^2+[2R_{ii}(0,0)-R_{ii}(-\boldsymbol{r}_0,0)-R_{ii}(\boldsymbol{r}_0,0)]\,\tau^2 \\ &\approx r_0^2\left[1+\frac{\epsilon}{3\nu}\tau^2\right],\end{aligned} \tag{40}$$

where ϵ is the energy-dissipation rate and ν is the kinematic viscosity. In the last step of (40) the relation

$$2R_{ii}(0,0)-R_{ii}(-\boldsymbol{r}_0,0)-R_{ii}(\boldsymbol{r}_0,0) \approx \boldsymbol{r}_0\boldsymbol{r}_0:\nabla\nabla\boldsymbol{R}(0,0) = r_0^2\frac{2}{3}\int_0^\infty \mathrm{d}k\,k^2E(k) = \frac{\epsilon}{3\nu}r_0^2$$

has been used, with $E(k) \equiv$ the energy spectrum. When the initial separation is small enough, (40) indicates that the two particles move almost together at short times.

In the long-time limit the relative displacement has a high probability of taking large values, so that the correlations $\boldsymbol{R}(\boldsymbol{y}'-\boldsymbol{\lambda}'-\boldsymbol{r}_0,\,\tau-\tau')$ and $\boldsymbol{R}(\boldsymbol{y}'+\boldsymbol{\lambda}'+\boldsymbol{r}_0,\,\tau-\tau')$ can be neglected in comparison with $\boldsymbol{R}(\boldsymbol{y}',\,\tau-\tau')$. By a Markovian approximation, which can be tolerated at long times, (39) leads to

$$\partial_\tau \bar{B}(\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0) \approx 2\boldsymbol{K}(\infty):\nabla_\lambda\nabla_\lambda\,\bar{B}(\boldsymbol{\lambda},\tau\,|\,\boldsymbol{r}_0,t_0), \tag{41}$$

where

$$\boldsymbol{K}(\infty) \equiv \int_0^\infty \mathrm{d}\tau'\,\boldsymbol{R}(\boldsymbol{y}',\tau')\,\bar{P}(\boldsymbol{y}',\tau')$$

is the asymptotic eddy-diffusivity tensor for a single particle. Equation (41) indicates that, in the long-time limit, the two particles move independently of each other. The second moment of the separation is

$$\langle \tilde{\boldsymbol{r}}(\tau)\, \tilde{\boldsymbol{r}}(\tau)\rangle \approx \langle \tilde{\boldsymbol{\lambda}}(\tau)\, \tilde{\boldsymbol{\lambda}}(\tau)\rangle \approx 2\boldsymbol{K}(\infty)\,\tau.$$

At intermediate times, we study the second moment of the relative displacement. It is found that

$$\frac{\mathrm{d}\langle \tilde{\boldsymbol{\lambda}}(\tau)\, \tilde{\boldsymbol{\lambda}}(\tau)\rangle}{\mathrm{d}\tau} = 2\int_0^\tau \mathrm{d}\tau' \iint \mathrm{d}\boldsymbol{y}'\, \mathrm{d}\boldsymbol{\lambda}' [2\boldsymbol{R}(\boldsymbol{y}', \tau-\tau') - \boldsymbol{R}(\boldsymbol{y}'-\boldsymbol{\lambda}'-\boldsymbol{r}_0, \tau-\tau') \\ -\boldsymbol{R}(\boldsymbol{y}'+\boldsymbol{\lambda}'+\boldsymbol{r}_0, \tau-\tau')]\, \bar{P}(\boldsymbol{y}', \tau-\tau')\, \bar{B}(\boldsymbol{\lambda}', \tau' \,|\, \boldsymbol{r}_0, t_0). \quad (42)$$

If the turbulence is also isotropic and r_0 is small enough, the Fourier transform of (42) yields

$$\frac{\mathrm{d}\sigma_\lambda^2(\tau)}{\mathrm{d}\tau} = \tfrac{8}{3}(2\pi)^3 \int_0^\tau \mathrm{d}\tau' \int_0^\infty \mathrm{d}k\, E(k, \tau-\tau')\, \bar{P}(k, \tau-\tau')\, [1-(2\pi)^3 B(k, \tau')]. \quad (43)$$

Batchelor (1952) and Obukhov (1959) suggested a Gaussian approximation for the relative dispersion and that $\bar{B}$ obeys the diffusion equation:

$$\partial_\tau \bar{B}(\boldsymbol{\lambda}, \tau) = \boldsymbol{K}^{\mathrm{r}}(\tau) : \nabla_\lambda \nabla_\lambda \bar{B}(\boldsymbol{\lambda}, \tau).$$

The eddy diffusivity $\boldsymbol{K}^{\mathrm{r}}(\tau)$ can be defined by Taylor's formula

$$\boldsymbol{K}^{\mathrm{r}}(\tau) = \frac{1}{2}\frac{\mathrm{d}\langle \tilde{\boldsymbol{\lambda}}(\tau)\, \tilde{\boldsymbol{\lambda}}(\tau)\rangle}{\mathrm{d}\tau},$$

and expressed explicitly by means of (42). It follows that

$$\frac{\mathrm{d}\sigma_\lambda^2(\tau)}{\mathrm{d}\tau} = \frac{8}{3}\int_0^\tau \mathrm{d}\tau' \int_0^\infty \mathrm{d}k\, E(k, \tau-\tau') \exp[-\tfrac{1}{2}k^2\sigma^2(\tau-\tau')]\{1-\exp[-\tfrac{1}{2}k^2\sigma_\lambda^2(\tau')]\}, \quad (44)$$

where $\sigma^2(\tau-\tau')$ and $\sigma_\lambda^2(\tau')$ are the one-dimensional variances of $\bar{P}(\boldsymbol{y}', \tau-\tau')$ and $\bar{B}(\boldsymbol{\lambda}', \tau')$ respectively. In a time period, eddies of large scales do not contribute much because of the factor $1-\exp[-\frac{1}{2}k^2\sigma_\lambda^2(\tau')]$. Eddies of the smallest scales do not contribute either, because of the factor $\exp[-\frac{1}{2}k^2\sigma^2(\tau-\tau')]$. These facts show that our equation preserves the characteristic of the relative dispersion that it is dominated by eddies of scales comparable to the effective separation. Note that $\boldsymbol{K}^{\mathrm{r}}(\tau)$ cannot be obtained by a memory cutoff of (38), unlike the case of a single particle.

For an analytical discussion, we write (44) as

$$\frac{\mathrm{d}^2\sigma_\lambda^2(\tau)}{\mathrm{d}\tau^2} = \frac{8}{3}\int_0^\infty \mathrm{d}k\, E(k)\{1-\exp[-\tfrac{1}{2}k^2\sigma_\lambda^2(\tau)]\} \\ -\frac{8}{3}\int_0^\infty \mathrm{d}k\, E(k)\{1-\exp[-\tfrac{1}{2}k^2\sigma^2(\tau)]\}\{1-\exp[-\tfrac{1}{2}k^2\sigma_\lambda^2(\tau^*)]\}, \quad (45)$$

where $0 < \tau^* < \tau$ and the mean-value theorem of integration has been applied. At intermediate times, the second term on the right-hand side of (45) is relatively unimportant, so that

$$\frac{\mathrm{d}^2\sigma_\lambda^2(\tau)}{\mathrm{d}\tau^2} \approx \frac{8}{3}\int_0^\infty \mathrm{d}k\, E(k)\{1-\exp[-\tfrac{1}{2}k^2\sigma_\lambda^2(\tau)]\}. \quad (46)$$

Based upon the Kolmogoroff spectrum $E(k) = C'\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}}$, $\sigma_\lambda^2(\tau)$ can easily be found. The result is

$$\sigma_\lambda^2(\tau) \approx 0.6065C'^{\frac{3}{2}}\epsilon\tau^3, \quad (47)$$

where C' is the Kolmogoroff constant. The eddy diffusivity for the relative dispersion is

$$K^{\mathrm{r}}(\tau) \equiv \tfrac{1}{3}K^{\mathrm{r}}_{ii}(\tau) = \frac{1}{2}\frac{\mathrm{d}\sigma^2_\lambda(\tau)}{\mathrm{d}\tau} \approx 1.8195 C'^{\frac{3}{2}}\epsilon\tau^2 \approx 1.2697 C'^{\frac{1}{2}}\epsilon^{\frac{1}{3}}[\sigma_\lambda(\tau)]^{\frac{4}{3}}. \tag{48}$$

Relations (47) and (48) correspond to Richardson's $\frac{4}{3}$-power law in Batchelor's (1952) version.

The Kolmogoroff spectrum is for the inertial subrange. It will be more interesting to test (44) with a spectrum for the whole range. In the numerical calculation, based upon (36) and (44), we have used the von Kármán spectrum and a spectrum smoothed from an experimental spectrum. The von Kármán spectrum is

$$E(k) = \frac{\mathscr{I}k^4}{[1+(k/k_{\mathrm{s}})^2]^{\frac{17}{6}}},$$

where $\mathscr{I}$ is the Loitsanski integral and k_{e} is a parameter with the dimension of a wavenumber. It has the advantage that both the empirical k^4 portion at small wavenumbers and the $k^{-\frac{5}{3}}$ portion at large wavenumbers are included. The semiempirical spectrum, in dimensionless form, is

$$E(k) = \begin{cases} 79.35k^4\exp(-5k) & (0 < k < 1.1429), \\ 0.6665k^{-3} & (1.1429 < k < 2.8571), \\ 0.08165k^{-1} & (2.8571 < k < 71.429), \\ 1.4056k^{-\frac{5}{3}} & (71.429 < k < 1079.1), \\ 326.61k^{-2.447} & (1079.1 < k < 28571), \\ 0 & (28571 < k). \end{cases}$$

This spectrum is smoothed from the v-component of spectra measured at 10 m height in the atmosphere by Mikkelsen (1983, figure 4.12*b*). The dimensional factors are $k_0 = 3.5\times10^{-3}\ \mathrm{m}^{-1}$ for k and $\langle\tilde{v}^2\rangle/2k_0$ for $E(k)$. For the two spectra both the variances of the displacement of a single particle and the relative displacement of a pair of particles are calculated. The results based upon the von Kármán spectrum are plotted in figure 2. The semiempirical spectrum is shown in figure 3, and the results based upon the semiempirical spectrum are shown in figure 4. The curves for $\sigma^2_\lambda(\tau)$ in the two figures both show a τ^3 slope, which corresponds to the $\frac{4}{3}$-power law.

Because of the nonlinearity of (39) and the dependence of $\overline{B}(\boldsymbol{\lambda}-\boldsymbol{\lambda}', \tau-\tau' \,|\, \boldsymbol{r}', t')$ on $(\boldsymbol{r}', t')$, the discussion of the relative dispersion can be made only with known models of the relative transition function. We have also applied (43) to the model suggested by Richardson (1926) and the one suggested by Okubo (1962). Both models have the diffusion equation

$$\partial_\tau \overline{B}(\lambda,\tau) = \frac{1}{\lambda^2}\frac{\partial}{\partial\lambda}\left[\lambda^2 K^{\mathrm{r}}(\lambda,\tau)\frac{\partial\overline{B}(\lambda,\tau)}{\partial\lambda}\right].$$

However, in Richardson's model

$$\overline{B}(\lambda,\tau) = a(\tau)\exp[-b(\tau)\,\tau)\,\lambda^{\frac{2}{3}}],$$

with

$$a(\tau) = \tfrac{8}{315}\pi^{-\frac{3}{2}}[b(\tau)]^{\frac{9}{2}}, \quad b(\tau) = \frac{1}{2}\left[\frac{429}{\sigma^2_\lambda(\tau)}\right]^{\frac{1}{3}}.$$

In Okubo's model

$$\overline{B}(\lambda,\tau) = a(\tau)\exp[-b(\tau)\,\lambda^{\frac{4}{3}}],$$

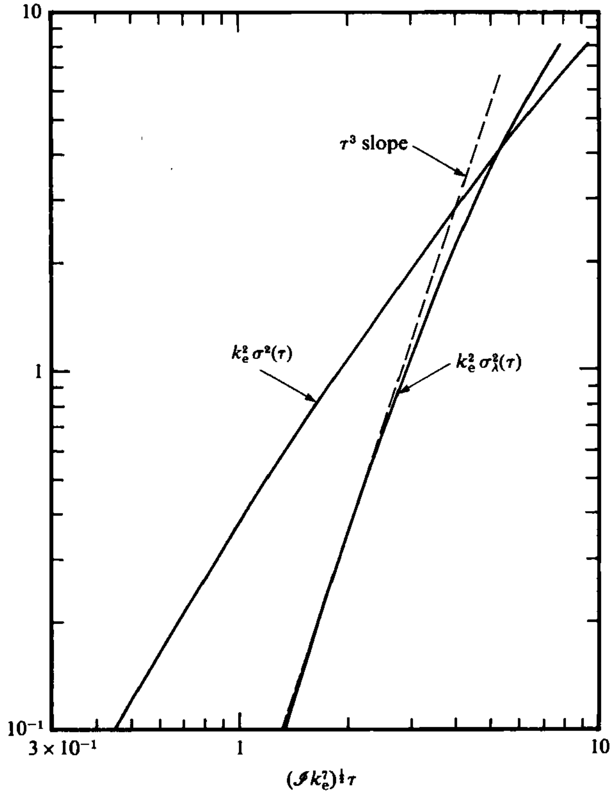

FIGURE 2. Variances of the displacements based upon the von Kármán spectrum.

with

$$a(\tau) = \frac{[b(\tau)]^{\frac{9}{4}}}{[3\pi \Gamma(\frac{9}{4})]},$$

$$b(\tau) = \left[\frac{20\Gamma(\frac{1}{4})}{77\Gamma(\frac{3}{4})}\right]^{-\frac{2}{3}} [\sigma_\lambda^2(\tau)]^{-\frac{2}{3}}.$$

The results, based upon the Kolmogoroff spectrum, are

$$\sigma_\lambda^2(\tau) \approx 0.5112 C'^{\frac{3}{2}} \epsilon \tau^3,$$

$$K^{\mathrm{r}}(\lambda, \tau) \approx 0.4771 C'^{\frac{1}{2}} \epsilon^{\frac{1}{3}} \tau^{\frac{4}{3}}$$

for Richardson's model, and

$$\sigma_\lambda^2(\lambda) \approx 0.5815 C'^{\frac{3}{2}} \epsilon \tau^3,$$

$$K^{\mathrm{r}}(\lambda, \tau) \approx 0.6576 C' \epsilon^{\frac{2}{3}} \lambda^{\frac{2}{3}} \tau$$

for Okubo's model. These results are comparable to those obtained by Kraichnan (1966), Lundgren (1981) and Misguich & Balescu (1982). It may be noted that, according to Sullivan's (1971) experiment, the Gaussian approximation seems better than Richardson's model. However, if Okubo's model is plotted on the same figure, it would show a better agreement with the experiment.

Although the Gaussian approximation has limitations, it may have practical applications. The behaviour of the relative dispersion in the short-time limit and in

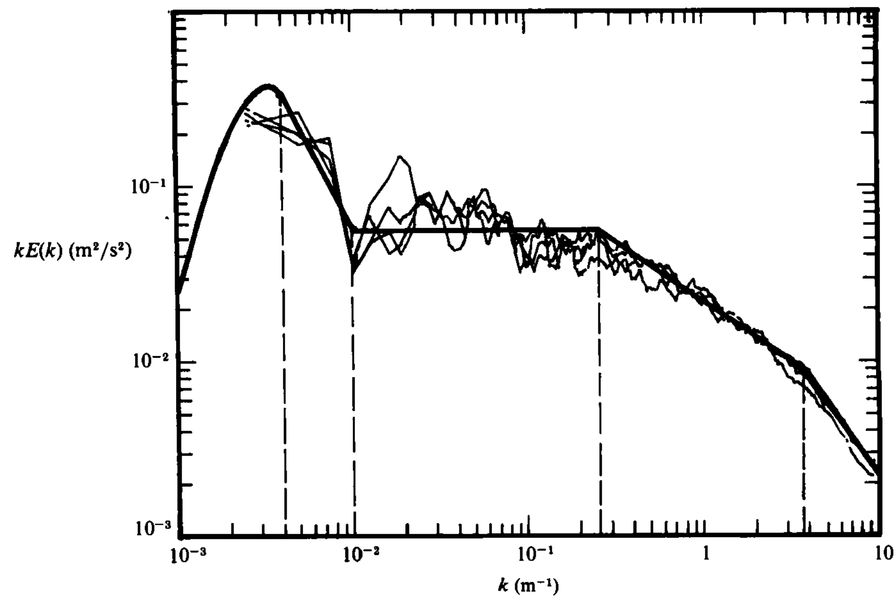

FIGURE 3. Semiempirical spectrum:

$$—\!—,\ kE(k) = \begin{cases} 1.0177\times 10^{14}k^{5}\exp(-5000k/3.5) & (0 < k \leqslant 0.004), \\ 5.5\times 10^{-6}k^{-2} & (0.004 < k \leqslant 0.01), \\ 5.5\times 10^{-2} & (0.01 < k \leqslant 0.25), \\ 0.02183k^{-\frac{2}{3}} & (0.25 < k \leqslant 3.777), \\ 0.06155k^{-1.447} & (3.777 < k \leqslant 100); \end{cases}$$

experimental v-spectrum (Mikkelsen 1983, figure 4.12*b*).

the long-time limit can be recovered by the Gaussian approximation, based upon (44). The discussion of fluid particles may apply to the diffusion of a passive scalar. The previous results, based upon the semiempirical spectrum, are shown (Jiang 1984) to be quite comparable to the experimental smoke-diffusion data that were obtained by Mikkelsen (1983) under conditions similar to those when the velocity spectra were measured.

4. Discussion and conclusion

Turbulent transport is always accompanied by the random motion of fluid elements. Because of this fact, the recently developed expansion methods have obtained remarkable successes in turbulence theory. The present paper uses the mean propagator for a renormalized expansion. The mean propagator is the deterministic part of the exact propagator which specifies the trajectory of the fluid particle. Therefore the intrinsic nonlinearity and the Lagrangian nature are included so that the expansion provides a good approximation at a low-order truncation when low-order moments of the displacements of particles are to be investigated. Both (32) and (39) are in real space and in terms of Eulerian velocity-correlation functions, so that they are convenient for possible applications.

In the derivation of the equations for the dispersion of a pair of marked fluid particles, the two-particle mean propagator has a meaning similar to that of the 'three-point Green function' introduced by Weinstock (1977) for the triple-velocity

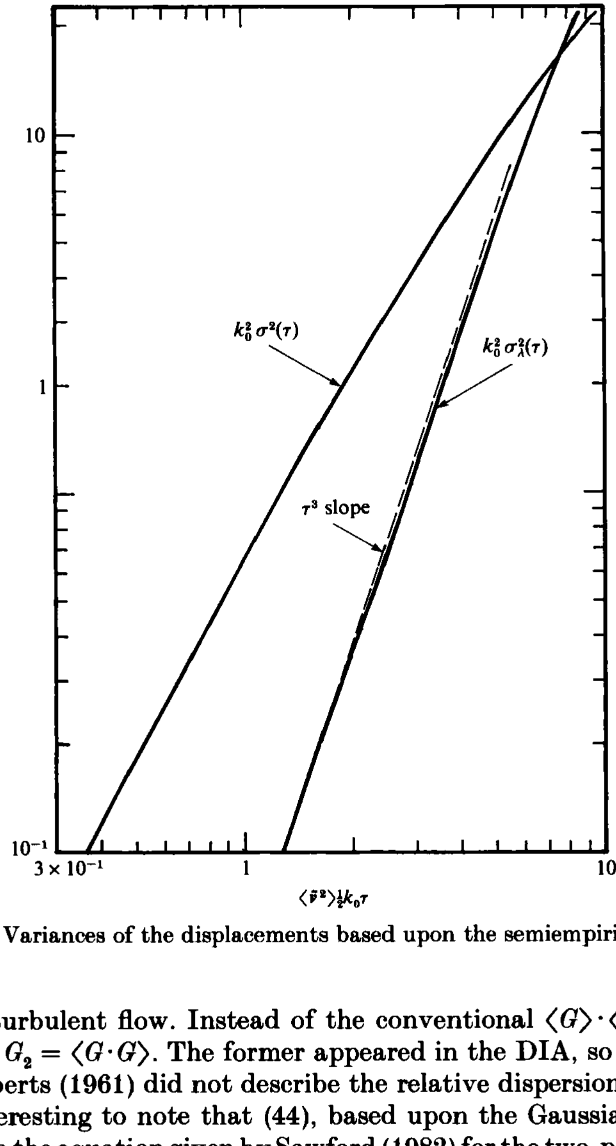

FIGURE 4. Variances of the displacements based upon the semiempirical spectrum.

correlation of turbulent flow. Instead of the conventional $\langle G\rangle\cdot\langle G\rangle$, ours is like a Green function $G_2 = \langle G\cdot G\rangle$. The former appeared in the DIA, so that the equation derived by Roberts (1961) did not describe the relative dispersion properly.

It is also interesting to note that (44), based upon the Gaussian approximation, is comparable to the equation given by Sawford (1982) for the two-particle Lagrangian velocity-correlation function and the one obtained by Mikkelsen (1982) for the relative diffusion of Gaussian puffs. However, our equation contains a time convolution which couples the variance of the relative displacement of the two particles with the variance of the displacement of a single particle. In view of the structure of the two-particle two-time Lagrangian velocity-correlation function, it seems more reasonable to have such a coupling.

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Appendix

The meaning of the exact propagator was examined by Weinstock (1969). Any physical quantity carried by the fluid particle, such as its mass, will obey an equation similar to (3):

$$[\partial_\tau + \hat{\mathrm{L}}(\tau)]\, F(\tau) = 0, \tag{A 1}$$

where the spatial dependence of F is implied. The present distribution of F can be related to its initial distribution by means of the exact propagator:

$$F(\tau) = \hat{U}(\tau, \tau_0)\, F(\tau_0). \tag{A 2}$$

When this expression is substituted into (A 1), we see that $\hat{U}$ obeys the equation

$$[\partial_\tau + \hat{\mathrm{L}}(\tau)]\, \hat{U}(\tau, \tau_0) = 0,$$

because both ∂_τ and $\hat{\mathrm{L}}(\tau)$ do not operate on $F(\tau_0)$. It is also obvious that $\hat{U}(\tau_0, \tau_0) = 1$, according to (A 2).

For the inhomogeneous equation, explicitly in $(\boldsymbol{y}, \tau)$-coordinates,

$$[\partial_\tau + \hat{\boldsymbol{v}}(\tau) \cdot \boldsymbol{\nabla}]\tilde{f}(\boldsymbol{y}, \tau) = \tilde{g}(\boldsymbol{y}, \tau), \tag{A 3}$$

the exact propagator can be shown to relate to a delta function. The Fourier transformation of (A 3) yields

$$[\partial_\tau + \mathrm{i}\boldsymbol{k} \cdot \hat{\boldsymbol{v}}(\tau)]\tilde{f}(\boldsymbol{k}, \tau) = \tilde{g}(\boldsymbol{k}, \tau). \tag{A 4}$$

Therefore

$$\begin{aligned} \tilde{f}(\boldsymbol{k}, \tau) &= \int_0^\tau \mathrm{d}\tau'\, \tilde{g}(\boldsymbol{k}, \tau') \exp\left[-\int_{\tau'}^{\tau} \mathrm{d}\tau''\, \mathrm{i}\boldsymbol{k} \cdot \hat{\boldsymbol{v}}(\tau'')\right] \\ &= \int_0^\tau \mathrm{d}\tau'\, \tilde{g}(\boldsymbol{k}, \tau') \exp\{-\mathrm{i}\boldsymbol{k} \cdot [\hat{\boldsymbol{Y}}(\tau) - \hat{\boldsymbol{Y}}(\tau')]\}, \end{aligned} \tag{A 5}$$

with the initial condition $\tilde{f}(\boldsymbol{k}, \tau_0)$ neglected as usual. However, (A 5) can be inversely transformed to give

$$\tilde{f}(\boldsymbol{y}, \tau) = \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}\boldsymbol{y}'\, \delta\{(\boldsymbol{y} - \boldsymbol{y}') - [\hat{\boldsymbol{Y}}(\tau) - \hat{\boldsymbol{Y}}(\tau')]\}\, \tilde{g}(\boldsymbol{y}', \tau'). \tag{A 6}$$

The exact propagator works as a Green function. Formally, (A 6) can also be written as

$$\tilde{f}(\tau) = \int_0^\tau \mathrm{d}\tau'\, \hat{U}(\tau, \tau')\, \tilde{g}(\tau'),$$

with the initial condition $\tilde{f}(\tau_0)$ neglected. We see that $\hat{U}(\tau, \tau')$ corresponds to the inner integration of (A 6) with respect to $\boldsymbol{y}'$. Recall the definition of the instantaneous transition function (2). Equation (A 6) is then written as

$$\tilde{f}(\boldsymbol{y}, \tau) = \int_0^\tau \mathrm{d}\tau' \int \mathrm{d}\boldsymbol{y}'\, \hat{P}(\boldsymbol{y} - \boldsymbol{y}', \tau - \tau' \mid \boldsymbol{x}', t')\, \tilde{g}(\boldsymbol{y}', \tau').$$

Therefore we have the equivalence

$$\hat{U}(\tau, \tau')\, \tilde{g}(\tau') = \int \mathrm{d}\boldsymbol{y}'\, \hat{P}(\boldsymbol{y} - \boldsymbol{y}', \tau - \tau' \mid \boldsymbol{x}', t')\, \tilde{g}\,(\boldsymbol{y}', \tau'),$$

$$\bar{U}(\tau, \tau')\, \tilde{g}(\tau') = \int \mathrm{d}\boldsymbol{y}'\, \bar{P}(\boldsymbol{y} - \boldsymbol{y}', \tau - \tau')\, \tilde{g}(\boldsymbol{y}', \tau').$$

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